Math 451: Introduction to General Topology
Lecture 19

Product top (= top. of pointwise convergence).

Recall that on the space $B(X,Y)$ of bounded functions from a set X to a metric space Y, we put the top. induced by the uniform metric. Uniform metric captures uniform convergence. This is good but not general enough, e.g. what if Y is a nonmetrizable top. space. More importantly, even when $Y=\mathbb{R}$, we have other interesting modes of convergence not captured by the uniform metric, namely, pointwise convergence: let $(f_n) \subseteq Y^X$ ($:=$ the set of all functions $X \to Y$) and $f \in Y^X$, we say that (f_n) converges pointwise to f if $f_n(x) \to f(x)$ for all $x \in X$. Even when $Y = \mathbb{R}$, pointwise convergence doesn't imply uniform convergence, by the following example.

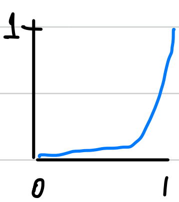
Example. Let $X := (0,1)$ and $Y := \mathbb{R}$, let $f_n : (0,1) \to \mathbb{R}$ be $f_n(x) := x^n$, so $(f_n) \subseteq C((0,1))$.

Claim 1. $f_n \to 0$ pointwise.

Proof. Fix $x \in (0,1)$. Then $x^n \to 0$ hence $\forall \varepsilon\ \forall^\infty n$ s.t. $x^n < \varepsilon$. $\square$

Claim 2. $f_n \not\to_u 0$, in fact (f_n) is not uniformly Cauchy.

Proof. We show $f_n \not\to_u 0$. $f_n \to_u 0 \iff \|f_n\|_u := \sup_{x\in(0,1)} |f_n(x)| \to 0$, which is not the case because $\forall n$, $\|f_n\|_u = 1$. Indeed, $\lim_{x\to 1} x^n = 1$, so $\sup_{x \in (0,1)} x^n = 1$. $\square$



We would like to define a topology on Y^X which exactly captures pointwise convergence, most importantly, on $\mathbb{R}^{[0,1]}$ or $\mathbb{R}^{\mathbb{R}}$. There is indeed such a topology, and when X is unctbl and Y is nontrivial, then this top is not metrizable (because it is not first ctbl); in particular, $\mathbb{R}^{[0,1]}$ is not metrizable. Since we clearly should care about ptwise convergence, we thus should care about nonmetrizable top. spaces. This top. is called the product topology (thinking of

Y^X as a product $\prod_{x\in X} Y$) or the top. of ptwise convergence. We begin with a product of finitely-many top. spaces.

<u>Finite products.</u> Let $X_0, \dots, X_{n-1}$ be top. spaces and consider $X := \prod_{i<n} X_i = X_0 \times X_1 \times \dots \times X_{n-1}$. The product top. on X is the one generated by open boxes, i.e. sets of the form $U_0 \times U_1 \times \dots \times U_{n-1}$, where $U_i \subseteq X_i$ open. This top is generated by one-base cylinders

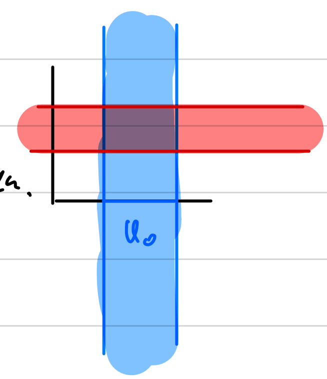
$$[i \mapsto U_i] := X_0 \times \dots \times X_{i-1} \times U_i \times X_{i+1} \times \dots \times X_{n-1},$$

because an open box $U_0 \times U_1 \times \dots \times U_{n-1}$ is a finite intersection of one-base cylinders:

$$U_0 \times U_1 \times \dots \times U_{n-1} = \bigcap_{i<n} [i \mapsto U_i].$$

Note that the intersection of two open boxes $U_0 \times \dots \times U_{n-1}$ and $V_0 \times \dots \times V_{n-1}$ is still an open box $(U_0 \cap V_0) \times \dots \times (U_{n-1} \cap V_{n-1})$, so the open boxes form a basis for the product top.

<u>Example.</u> In $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, a base-one open cylinder looks like this



These are the sets $\mathbb{R} \times V$ and $U \times \mathbb{R}$, where $U, V \subseteq \mathbb{R}$ are open. Open boxes are sets $U \times V = (\mathbb{R} \times V) \cap (U \times \mathbb{R})$, as in the picture.

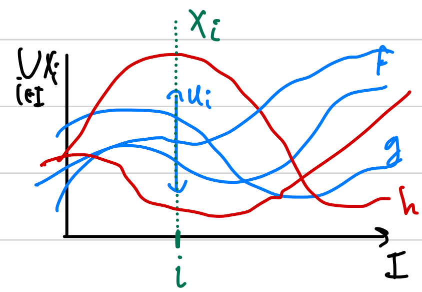
<u>Remark.</u> Since open boxes in $\mathbb{R}^n$ generate the Euclidean top of $\mathbb{R}^n$ (i.e. the one induced by the metric $d_2(x,y) := \left(\sum |x(i) - y(i)|^2\right)^{\frac{1}{2}}$), the Euclidean top of $\mathbb{R}^n$ is just the product top coming from the usual top on $\mathbb{R}$.

<u>Remark.</u> In $X = X_0 \times \dots \times X_{n-1}$, the one-base cylinders $[i \mapsto U_i]$ are exactly the sets $\mathrm{proj}_i^{-1}(U_i)$, where $\mathrm{proj}_i : X \to X_i$, defined by $x \mapsto x(i)$, is the projection onto the ith coordinate, also called evaluation at i. Thus, the product top is exactly the top. generated by the projection functions proj_i, $i<n$, i.e. it is the smallest top. on X making the projections continuous.

<u>Arbitrary products.</u> Let I be an arbitrary index set, e.g. $I = \mathbb{N}$ or $\mathbb{R}$. For each $i \in I$,

Let X_i be a top space. Consider the product $X := \prod_{i\in I} X_i := \{ f \in (\bigcup_{i\in I} X_i)^I : \forall i \in I \; f(i) \in X_i \}$.
Recall that the statement "product of nonempty sets is nonempty" is equivalent to AC.
The product top on X is generated by one-base cylinders:

$$[i \mapsto U_i] := \{ f \in X : f(i) \in U_i \}.$$



Thinking of X as a set of functions $f : I \to \bigcup_{i\in I} X_i$, we can draw the one-base cylinder $[i \mapsto U_i]$ as follows: it is all the blue functions which go through the hoop U_i at the i^{th} coordinate.
The finite intersections of one-base cylinders are finite-base cylinders:

$$[i_1 \mapsto U_{i_1}, i_2 \mapsto U_{i_2}, \dots, i_k \mapsto U_{i_k}] := \bigcap_{j=1}^{k} [i_j \mapsto U_{i_j}].$$

We simply call these sets cylinders.
Cylinders form a basis, being the finite intersection of sets of a prebasis.

Example. For a nonempty Σ, recall that $\Sigma^{\mathbb{N}}$ is equipped with the top coming from the metric $d(x,y) := \begin{cases} 2^{-n(x,y)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$, where $n(x,y) :=$ the least $n \in \mathbb{N}$ s.t. $x(n) \neq y(n)$.

The top induced by this metric has the cylinders $[w]$ as its basis, where for $w \in \Sigma^{<\mathbb{N}}$,

$$[w] := \{ x \in \Sigma^{\mathbb{N}} : \forall i < \mathrm{lh}(w) \;\; x(i) = w(i) \}.$$

Now equip Σ with the discrete top., so every singleton $\{\sigma\} \subseteq \Sigma$ is open. Then

$$[w] = [0 \mapsto \{w(0)\}, 1 \mapsto \{w(1)\}, \dots, n-1 \mapsto \{w(n-1)\}],$$

where $n := \mathrm{lh}(w)$. Thus, the metric-open sets are open in the product top on $\Sigma^{\mathbb{N}}$ with a discrete Σ. Conversely, observe that every one-base cylinder $[i \mapsto U_i]$ is metric open because it is a union of cylinders of the form $[w]$ for $w \in \Sigma^{<\mathbb{N}}$ (i.e. balls in the metric), namely: $[i \mapsto U_i] = \bigcup_{v \in \Sigma} \bigcup_{\sigma \in U_i} [v\sigma]$. Thus, the metric top is equal to the product top. on $\Sigma^{\mathbb{N}}$.

Remark. Again let $\mathrm{proj}_i : X \to X_i$ by $f \mapsto f(i)$ be the projection onto the i^{th} coordinate or evaluation at i. Note that that $[i \mapsto U_i] = \mathrm{proj}_i^{-1}(U_i)$, where $i \in I$, $U_i \subseteq X_i$ open.
Thus, the product top on X is exactly the top generated by the projections.